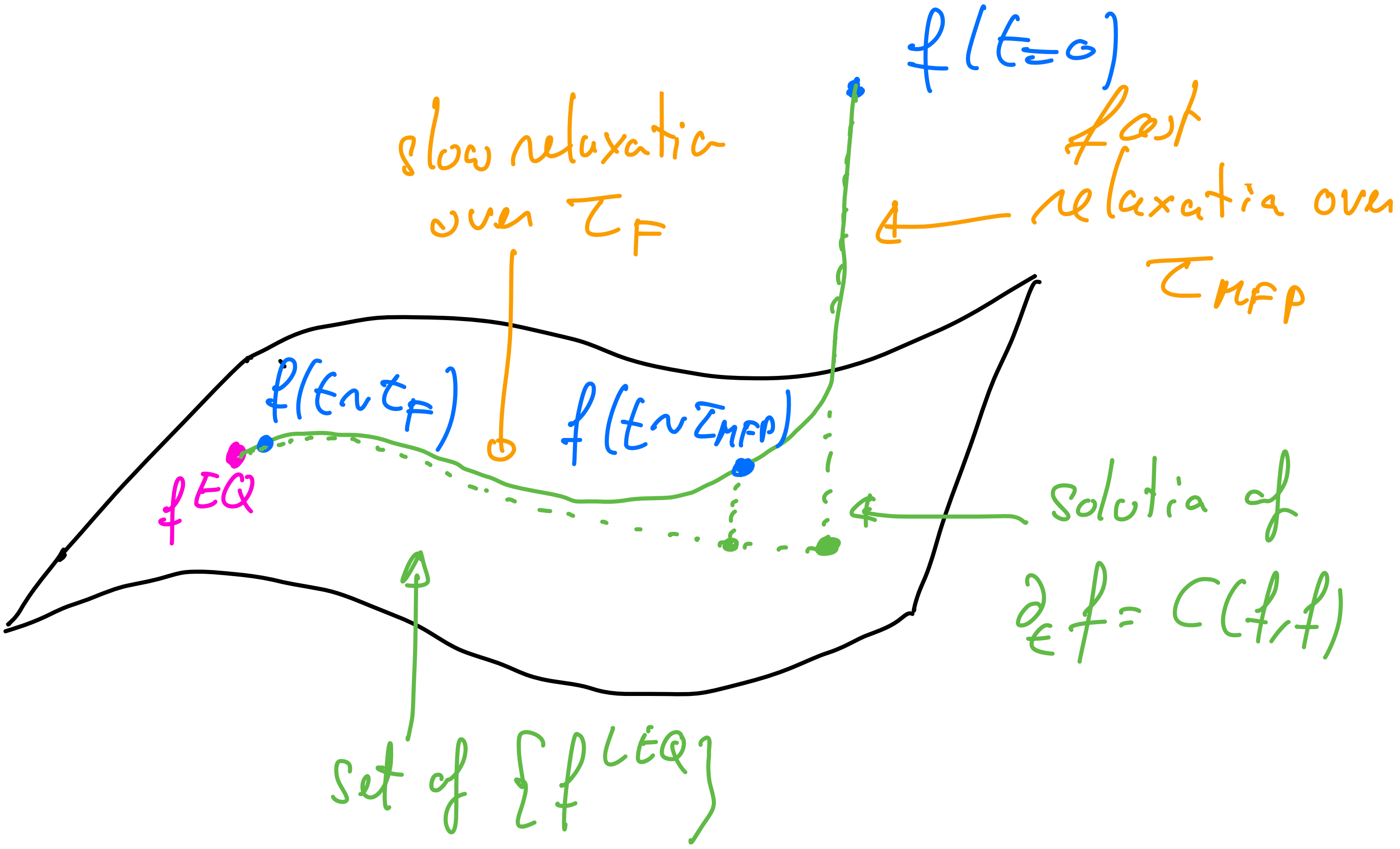


Perturbation theory

$$f(\vec{q},\vec{p},t) = f_0(\vec{q},\vec{p},t) + \varepsilon f_1(\vec{q},\vec{p},t)$$

Hydrodynamic fields

$$\int f\,dp = n(q)$$

$$\int \vec{v} f\,dp = n(\vec{q})\,\vec{u}(q) \quad : \quad \vec{v} = \frac{\vec{p}}{m}$$

$$\int \frac{M}{2}(\vec{v}-\vec{u})^2 f\,dp = n(\vec{q})\,\varepsilon(\vec{q}) = \frac{3}{2} n(\vec{q})\,k_B T(\vec{q})$$

leading order (BE)

$$\hat{C}(f,f)=0 \Rightarrow \quad f_0(\vec{q},\vec{p},t) = \frac{n}{(2\pi M k_B T)^{3/2}}\, e^{-\frac{M\,\delta\vec{v}^{\,2}}{2kT}}$$

$$\delta\vec{v}(\vec{q},\vec{p},t) = \vec{v} - \vec{u}(\vec{q},t)$$

Hydrodynamic equations

$$\boxed{D_t n \equiv \left(\partial_t + u_\alpha \partial_{q_\alpha}\right) n = - n\, \partial_{q_\alpha} u_\alpha}$$

$$\boxed{M\, D_t u_\alpha = -\frac{1}{n}\, \partial_{q_\beta} \cdot P_{\alpha\beta} \Leftrightarrow M\, D_t \vec{u} = -\frac{1}{n} \vec{\nabla} \cdot \bar{\bar{P}}}$$

with $P_{\alpha\beta} = Mn \langle \delta v_\alpha\, \delta v_\beta \rangle$

$$\boxed{\partial_t T + u_\alpha \partial_\alpha T = -\frac{2}{3nk_B}\, \partial_\alpha h_\alpha - \frac{2}{3nk_B}\, P_{\alpha\beta}\, u_{\alpha\beta}}$$

where

- $u_{\alpha\beta} = \frac{1}{2}\left(\partial_{q_\alpha} u_\beta + \partial_{q_\beta} u_\alpha\right)$ is the strain rate tensor
- $h_\alpha = \frac{nM}{2} \langle \delta v_\alpha\, \delta v_\beta \delta v_\beta \rangle$ is the heat flux

Closure: To compute the evolution of $n, T, \vec{u}$, we need to compute $P_{\alpha\beta} = nM \langle \delta v_\alpha \delta v_\beta \rangle$ & $h_\alpha = \frac{nM}{2} \langle \delta v_\alpha \delta v_\beta \delta v_\beta \rangle$, which we can do perturbatively using f:

$$n, T, \vec{u} \Rightarrow f_0 \Rightarrow P^0_{\alpha\beta}, h^0_\alpha \Rightarrow \partial_t (n, T, \vec{u})^0 \Rightarrow \partial_t f_0$$

Pressure & heat flux

$P^0_{\alpha\beta} = n\, M \langle \delta v_\alpha\, \delta v_\beta \rangle_0 = n\, k_B T\, \delta_{\alpha\beta}$ ideal gas law

$h^0_\alpha = 0$ because it involves an odd moment of δv

leading order dynamics

$$\partial_t n + u_\alpha \partial_{q_\alpha} n = -n\, \partial_{q_\alpha} u_\alpha \qquad (1)$$

$$m\, \partial_t u_\alpha + m\, u_\beta \partial_{q_\beta} u_\alpha = -\frac{1}{n} \partial_{q_\alpha} [n\, kT] \qquad (2)$$

$$\partial_t T + u_\alpha \cdot \partial_\alpha T = -\frac{2}{3} T\, \partial_{q_\alpha} u_\alpha \qquad (3)$$

Q: Do these equations lead f_0 to f^{EQ}?

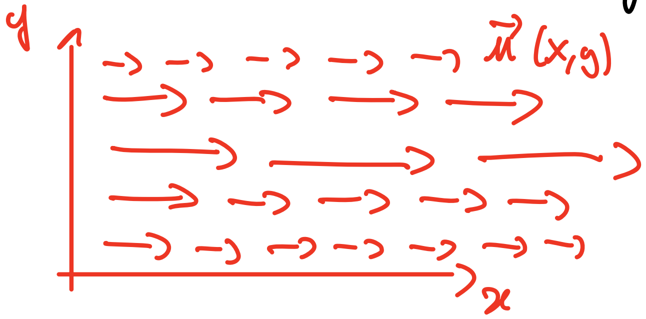
No relaxation of shear modes

Let us consider an initial condition in which the gas is sheared:

$n_0 = \bar{n}$ uniform

$T_0 = \bar{T}$ uniform

$\vec{u} = u(y)\, \vec{e}_x$



In eq (2) ; $\vec{\nabla} \cdot (n_0 T_0) = 0$ & $\vec{u} \cdot \vec{\nabla} \vec{u} = u(y)\, \partial_x u(y)\, \vec{e}_x = 0$

$\Rightarrow \partial_t \vec{u}_0 = 0$ & $\vec{u}_0$ is NOT relaxing to $\vec{v} = 0$ ...

At order ε^0, f^{LEQ} does not necessary relax to f^{EQ} $\Rightarrow$ higher order!

2.4.4) First-order hydrodynamics

Single time approximation : $\partial_t f = C(f,f)$ makes f relax to f_0 in a time $\sim \tau_{MFP}$, thanks to a complicated superposition of processes involving time scales spanning $[\tau_{col}, \tau_{MFP}]$.

Seen from a larger time, we approximate this as a single relaxation process: $\partial_t f \simeq -\frac{1}{\tau_{MFP}}(f - f_0)$

$$\Rightarrow C(f,f) \simeq -\frac{1}{\tau_{MFP}}(f-f_0) = -\frac{1}{\tau_{MFP}}\left(\varepsilon f_1 + \varepsilon^2 f_2 + \dots\right) \quad \text{where } \varepsilon = \frac{\tau_{MFP}}{\tau_F}$$

$$\simeq -\frac{1}{\tau_F} f_1$$

Rescaled dynamics $t = \tau_F \hat{t}$

$$\partial_{\hat{t}} f = -L_F f + \frac{1}{\varepsilon}\hat{C}(f,f) \quad \text{with } \hat{C}(f,f) = \tau_{MFP}\, C(f,f) \simeq -\varepsilon f_1$$

$$\Rightarrow \partial_{\hat{t}} f = -L_F f - f_1 \quad ; \quad f = f_0 + \varepsilon f_1$$

$$\Rightarrow \mathcal{O}(\varepsilon^0) : \quad \partial_{\hat{t}} f_0 = -L_F f_0 - f_1$$

$$\Rightarrow f_1 = \underbrace{-L_F f_0 - \partial_{\hat{t}} f_0}_{\text{known from leading order hydro}} = -\tau_F \underbrace{\left(\vec{v}\cdot\partial_{\vec{q}} f_0 + \partial_t f_0\right)}_{\text{easier than working with } \hat{t} \text{ to use hydrodynamic equations}}$$

Finally, we write $f = f_0 + \varepsilon f_1 = f_0 (1 + g_1)$

with $g_1 = -\tau_{MFP} \left(\partial_t \ln f_0 + \vec{v} \cdot \partial_{\vec{q}} \ln f_0 \right)$ that we need to determine to leading order in $\varepsilon \Rightarrow$ use 0th order hydrodynamics for the right-hand side.

Logic:

$$O\left(\frac{1}{\varepsilon}\right) \Rightarrow f_0 \Rightarrow h_0, P^0_{\alpha\beta} \Rightarrow n_0, T_0, u_0 \overset{O(1)}{\Rightarrow} f_1 \Rightarrow h_1, P^1_{\alpha\beta} \Rightarrow n_1, T_1, u_1$$

First order phase-space density:

Using $f_0 = \frac{n_0}{(2\pi m k_B T_0)^{3/2}} e^{-\frac{m \delta \vec{v}^2}{2 k T_0}}$ leads to

$$g_1 = -\tau_{MFP} \left(\partial_t + v_\alpha \partial_{q_\alpha} \right) \left[\ln n_0 - \frac{3}{2} \ln T_0 - \frac{m \delta \vec{v}^2}{2 k_B T_0} \right]$$

Using

$$D_t n_0 = -n_0 \partial_{q_\alpha} u^0_\alpha$$

$$m D_t u^0_\alpha = -\frac{1}{n_0} \partial_{q_\alpha} \left[n_0 k_B T_0 \right]$$

$$D_t T_0 = -\frac{2}{3} T_0 \partial_{q_\alpha} u^0_\alpha$$

leads to

$$\boxed{g_1 = -\frac{\tau_{MFP}\, m}{k_B T_0} u^0_{\alpha\beta} \left(\delta v_\alpha \delta v_\beta - \frac{\delta \vec{v}^2}{3} \delta_{\alpha\beta} \right) - \frac{\tau_{MFP}}{T_0} \left(\partial_{q_\alpha} T_0 \right) \left(\frac{m}{2 k_B T_0} \delta \vec{v}^2 - \frac{5}{2} \right)}$$

Note that $g_1(T_0, u^0) = g_1(T, u) + O(\varepsilon) \Rightarrow$ essentially equivalent to this order

Proof: Drop the "0" sub & superscript & use

$$\partial_t + v_\alpha\partial_{q_\alpha} = D_t + (v_\alpha - u_\alpha)\partial_\alpha = \overset{①}{D_t} + \overset{②}{\delta v_\alpha\partial_{q_\alpha}}$$

$$-\frac{g_1}{\tau_{MFP}} = \frac{1}{n}\left(D_t + \delta v_\alpha\partial_{q_\alpha}\right)n - \frac{3}{2T}\left(D_t + \delta v_\alpha\partial_{q_\alpha}\right)T - \frac{m}{2k_B}\left(D_t + \delta v_\alpha\partial_{q_\alpha}\right)\frac{\delta\vec{v}^2}{T}$$

$$= ①:\ \underbrace{-\partial_{q_\alpha}u_\alpha - \frac{3}{2}\left(-\frac{2}{3}\partial_{q_\alpha}u_\alpha\right)}_{=0} - \frac{m}{\cancel{2}k_B}\cancel{2}\delta v_\beta\frac{D_t(-u_\beta)}{T} + \frac{m}{2k_B}\frac{\delta\vec{v}^2}{T^2}D_t T$$

$$② + \frac{\delta v_\alpha}{n}\partial_{q_\alpha}n - \frac{3\delta v_\alpha}{2T}\partial_{q_\alpha}T - \frac{m\delta v_\alpha}{2k_B}\left(\frac{2\delta v_\beta}{T}\partial_{q_\alpha}(-v_\beta) - \frac{\delta\vec{v}^2}{T^2}\partial_{q_\alpha}T\right)$$

$$= \frac{\cancel{m}}{k_BT}\delta v_\beta\left(-\frac{1}{n\cancel{m}}\partial_{q_\beta}(n k_B T)\right) - \frac{m}{2k_B}\frac{\delta\vec{v}^2}{T^2}\frac{2}{3}T\partial_{q_\alpha}u_\alpha$$

$$\overset{0}{\updownarrow}$$

$$+\frac{\delta v_\alpha}{n}\partial_{q_\alpha}n - \frac{\delta v_\alpha}{T}\partial_{q_\alpha}T\left(\frac{3}{2} - \frac{m\delta\vec{v}^2}{2k_BT}\right) + \frac{m}{k_BT}\underbrace{\delta v_\alpha\delta v_\beta\partial_{q_\alpha}u_\beta}_{=\delta v_\alpha\delta v_\beta u_{\alpha\beta}}$$

$$= -\frac{\delta v_\beta}{T}\partial_{q_\beta}T - \frac{m}{3k_BT}\delta\vec{v}^2\underbrace{\partial_{q_\alpha}u_\alpha}_{u_{\alpha\beta}\delta_{\alpha\beta}} - \frac{\delta v_\alpha}{T}\partial_{q_\alpha}T\left(\frac{3}{2} - \frac{m}{2}\frac{\delta\vec{v}^2}{k_BT}\right) + \frac{m}{k_BT}\delta v_\alpha\delta v_\beta u_{\alpha\beta}$$

$$g_1 = -\tau_{MFP}\frac{m}{k_BT}u_{\alpha\beta}\left[\delta v_\alpha\delta v_\beta - \frac{\delta\vec{v}^2}{3}\delta_{\alpha\beta}\right] + \tau_{MFP}\frac{\delta v_\alpha}{T}\partial_{q_\alpha}T\left(\frac{5}{2} - \frac{m\delta\vec{v}^2}{2k_BT}\right)$$

First order pressure & heat flux:

$$f = f_0(1+g_1) \Rightarrow \langle O\rangle_f = \langle O(1+g_1)\rangle_{f_0}$$

$$\Rightarrow P_{\alpha\beta} = n\,m\langle\delta v_\alpha\delta v_\beta\rangle = n\,m\langle\delta v_\alpha\delta v_\beta\rangle_0 - n\,m^2\frac{\tau_{MFP}}{k_BT}u_{\gamma\delta}\Big[$$

$$\langle\delta v_\alpha\delta v_\beta\delta v_\gamma\delta v_\delta\rangle - \frac{\delta_{\gamma\delta}}{3}\langle\delta v_\alpha\delta v_\beta\delta\vec{v}^2\rangle\Big] + \underset{\substack{\uparrow\\ \langle\delta v\ldots\delta v\rangle\\ \text{odd \# of } \delta v}}{0}$$

Wich theorem

Moments of Gaussians are given by the sum over all possible unitary pairings.

$$\langle \delta v_\alpha \delta v_\beta \delta v_\gamma \delta v_\delta \rangle_0 = \frac{(k_B T)^2}{m^2}\left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta}\right)$$

$$\Rightarrow u_{\gamma\delta}\langle \delta v_\alpha \delta v_\beta \delta v_\gamma \delta v_\delta \rangle = \frac{(k_B T)^2}{m^2}\left(u_{\gamma\gamma}\delta_{\alpha\beta} + 2u_{\alpha\beta}\right)$$

$$\langle \delta v_\alpha \delta v_\beta \delta v_\gamma \delta v_\gamma \rangle = \frac{(k_B T)^2}{m^2}\left(3\delta_{\alpha\beta} + 2\underbrace{\delta_{\alpha\gamma}\delta_{\beta\delta}}_{\delta_{\alpha\beta}}\right) = 5\delta_{\alpha\beta}\frac{(k_B T)^2}{m^2}$$

$$\Rightarrow P_{\alpha\beta} = n k_B T \delta_{\alpha\beta} - n k_B T \tau_{MFP}\left(2u_{\alpha\beta} + u_{\gamma\gamma}\delta_{\alpha\beta} - \frac{5}{3}u_{\gamma\gamma}\delta_{\alpha\beta}\right)$$

$$P_{\alpha\beta} = n k_B T\left[\delta_{\alpha\beta}\left(1 + \frac{2}{3}\tau_{MFP}u_{\gamma\gamma}\right) - 2\tau_{MFP}u_{\alpha\beta}\right]$$

Relaxation of shear flow:

$\vec{u} = u(y)\,\vec{e}_x$

$$u_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha\right) = \frac{1}{2}\left(\delta_{\alpha y}\delta_{\beta x} + \delta_{\beta y}\delta_{\alpha x}\right)u'(y)$$

$$P_{\alpha\beta} = n k_B T \delta_{\alpha\beta} - n k_B T \tau_{MFP}\left(\delta_{\alpha y}\delta_{\beta x} + \delta_{\beta y}\delta_{\alpha x}\right)u'(y)$$

$$m D_t u_x = -\frac{1}{n}\partial_{q_\alpha}P_{\alpha x}$$

$$\Leftrightarrow m\partial_t u(y) + m\, u_x \cdot \underbrace{\partial_x u(y)}_{=0} + m\underbrace{u_y \cdot \partial_y}_{=0} u(y) = -\frac{1}{n}\partial_y\left(-n k_B T \tau_{MFP} u'(y)\right)$$

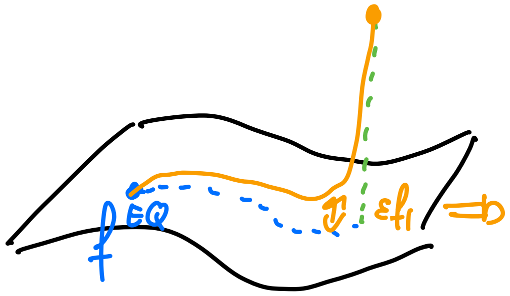
For $n = n_0$, $T = T_0$ $\Rightarrow$ $\partial_t u(y,t) = \underbrace{\frac{\tau_{MFP} k_B T}{m}}_{\nu \text{ is the kinematic viscosity}} \partial_{yy} u(y,t)$ diffusion equation

$\Rightarrow$ $u(y)$ relaxes to $u=0$ $\Rightarrow$ equilibration!

Relaxation mechanism

(i) f_1 alters the statistics of f, and thus $P_{\alpha\beta}$ & h_α

(ii) Transport makes the hydrodynamic field relax



$$\Rightarrow P_{\alpha\beta} = n k_B T \delta_{\alpha\beta} + \underbrace{\delta P}_{\text{drives relaxation}}$$

Heat conductivity:

One can use similar computations to find the heat flux $h_\alpha = -\kappa \partial_\alpha T$

where κ is the thermal conductivity $\kappa = \frac{5}{2} \frac{n k_B^2 T}{m} \tau_{MFP}$

Proof:

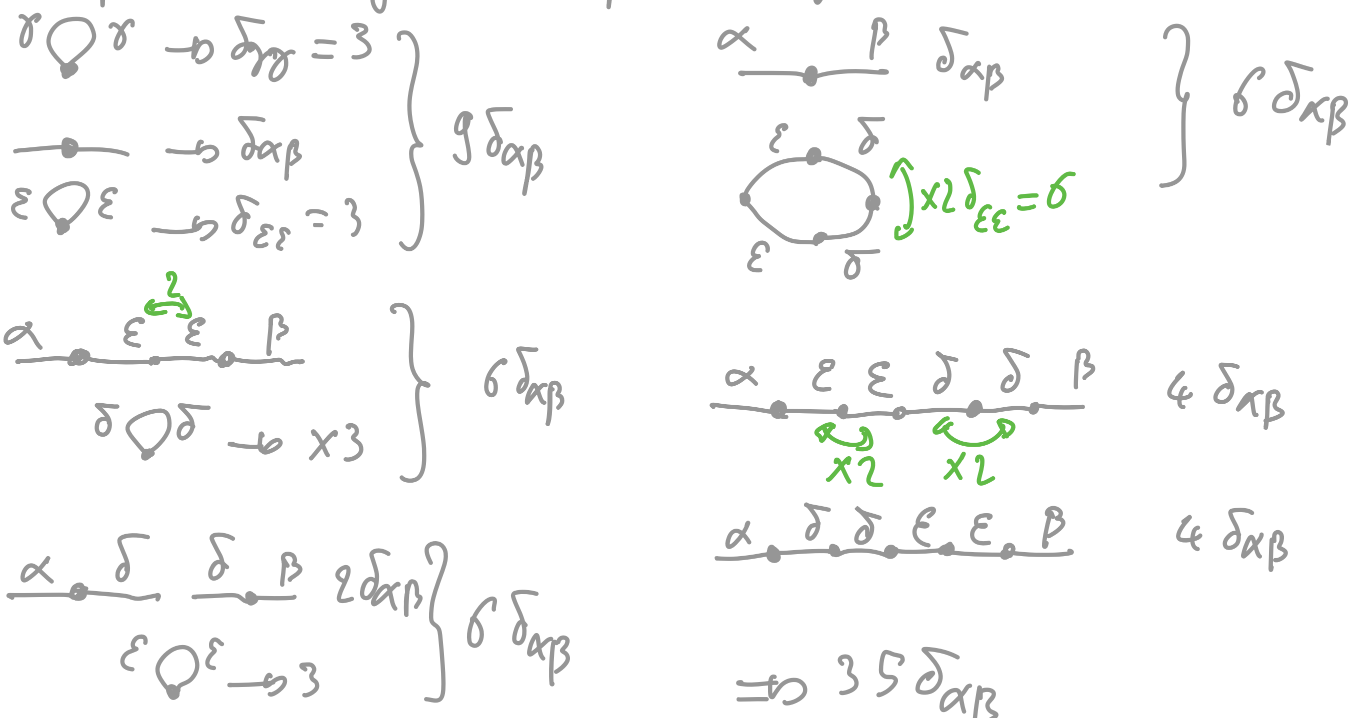
$* \; h_\alpha = \frac{n M}{2} \langle \delta v_\alpha \delta \vec{v}^2 (1+g_1) \rangle_0$ → only even powers matter

→ — odd ⟶ in g_1

$$h_\alpha = -\frac{n M \tau_{MFP}}{2T} (\partial_{q_\beta} T) \left\langle \frac{M}{2k_BT} (\delta \vec{v}^2)^2 \delta v_\beta \delta v_\alpha - \frac{5}{2} \delta \vec{v}^2 \delta v_\alpha \delta v_\beta \right\rangle$$

$*$ We need to compute $\langle \delta v_\gamma \delta v_\gamma \delta v_\varepsilon \delta v_\varepsilon \delta v_\alpha \delta v_\beta \rangle = \frac{k^3 T^3}{M^3} \delta_{\alpha\beta} \times$ (combinatorial factor)

All possible pairings ① Graph strategy



⇒ $35\,\delta_{\alpha\beta}$

② Math strategy. Sum only if $\sum_i$ explicit.

$$\sum_{\varepsilon,\delta} \langle \delta v_\alpha \delta v_\beta \delta v_\varepsilon \delta v_\varepsilon \delta v_\gamma \delta v_\gamma \rangle = \delta_{\alpha\beta} \langle \delta v_\alpha^2 \sum_\varepsilon \delta v_\varepsilon^2 \sum_\delta \delta v_\delta^2 \rangle$$

$$= \delta_{\alpha\beta} \Big[\langle \delta v_\alpha^6 \rangle_{\varepsilon=\delta=\alpha} + 2 \sum_{\varepsilon\neq\alpha} \langle \delta v_\alpha^4 \, \delta v_\varepsilon^2 \rangle_{\delta=\alpha}$$

$$+ \sum_{\varepsilon\neq\alpha} \langle \delta v_\alpha^2 \, \delta v_\varepsilon^4 \rangle + \sum_{\varepsilon\neq\delta\neq\alpha} \langle \delta v_\alpha^2 \, \delta v_\varepsilon^2 \, \delta v_\delta^2 \rangle \Big]$$

$$= \delta_{\alpha\beta} \Big[5 \langle \delta v_\alpha^2 \rangle \langle \delta v_\alpha^4 \rangle + 4 \langle \delta v_\alpha^4 \rangle \langle \delta v_\varepsilon^2 \rangle$$

$$+ 2 \langle \delta v_\alpha^2 \rangle \langle \delta v_\varepsilon^4 \rangle + 2 \langle \delta v_\alpha^2 \rangle \langle \delta v_\varepsilon^2 \rangle \langle \delta v_\gamma^2 \rangle \Big]$$

$$= \delta_{\alpha\beta} \left[15 \left(\frac{k_BT}{M}\right)^3 + 12 \left(\frac{k_BT}{M}\right)^3 + 6 \left(\frac{k_BT}{M}\right)^3 + 2 \left(\frac{k_BT}{M}\right)^3 \right]$$

$$= 35 \left(\frac{k_BT}{M}\right)^3 \delta_{\alpha\beta}$$

$$h_\alpha = -\frac{n M \tau_{MFP}}{2T}\left(\partial_{q_\beta} T\right)\left\langle \frac{M}{2kT}\left(\delta \vec{v}^{\,2}\right)^2 \delta v_\beta \delta v_\alpha - \frac{5}{2}\delta \vec{v}^{\,2}\delta v_\alpha \delta v_\beta \right\rangle$$

$$= -\frac{n M \tau_{MFP}}{4T}\left(\partial_{q_\beta} T\right)\delta_{\alpha\beta}\left(\frac{M}{kT}\cdot 35\left(\frac{k_B T}{M}\right)^3 - 25\left(\frac{k_B T}{M}\right)^2\right)$$

$$\boxed{h_\alpha = -\frac{5}{2}\frac{n}{M}\tau_{MFP}\, k_B^2\, T\left(\partial_{q_\alpha} T\right)}$$

First-order hydrodynamics

$$D_t n = -n\, \partial_{q_\alpha} u_\alpha$$

$$m D_t u_\alpha = -\frac{1}{n}\partial_{q_\beta} \cdot P_{\alpha\beta} = -\frac{1}{n}\partial_{q_\alpha}\left[n k_B T + \frac{2}{3} n k_B T \tau_{MFP}\partial_\gamma v_\gamma\right)$$

$$+\frac{1}{n}\partial_{q_\beta}\left[n k_B T\, \tau_{MFP}\, u_{\alpha\beta}\right]$$

$$D_t T = \frac{5}{3 n k_B}\partial_{q_\alpha}\left[\frac{n}{m}\tau_{MFP} k_B^2 T \partial_{q_\alpha} T\right] - \frac{2T}{3}\left[\partial_{q_\alpha} u_\alpha - \tau_{MFP} u_{\alpha\beta} u_{\alpha\beta} + \frac{2}{3}\tau_{MFP}\left(\partial_\alpha v_\alpha\right)^2\right]$$

Relaxation to equilibrium

$n = n_0 + \delta n$; $\vec{u} = \delta\vec{u}$; $T = T_0 + \delta T$

linearized dynamics for $\begin{pmatrix}\delta n \\ \delta\vec{u} \\ \delta T\end{pmatrix}$: $\partial_t \begin{pmatrix}\delta n \\ \delta\vec{u} \\ \delta T\end{pmatrix} = M \cdot \begin{pmatrix}\delta n \\ \delta\vec{u} \\ \delta T\end{pmatrix}$

The eigenvalues of M have negative real parts, leading to the decay of the perturbation & the convergence to equilibrium ($\vec{u}=0$, $T=T_0$ & $n=n_0$).

Few more thoughts

We set up a nice perturbation for f & then came back to t when computing hydrodynamic equations, which is frustrating. A longer but neater route consist in rescaling also the hydro-dynamic time & mods.

To $t = \hat{t}\,\tau_F$ correspond $\boxed{\vec{p} = \frac{1}{\tau_F}\hat{\vec{p}}}$

$\hat{n} = n$ so that $\hat{f}\,d\hat{\vec{p}} = f\,d\vec{p}$ & $\boxed{f = \tau_F \hat{f}.}$

$$\vec{u} = \int dp\, f\, \vec{p} = \frac{1}{\tau_F}\int d\hat{p}\,\hat{f}\,\hat{\vec{p}} \Rightarrow \boxed{\vec{u} = \frac{1}{\tau_F}\hat{\vec{u}}}$$

Similarly, $\boxed{T = \frac{1}{\tau_F}\hat{T}\ ,\ D_t = \frac{1}{\tau_F}\hat{D}_t\ ,\ P_{\alpha\beta} = \frac{1}{\tau_F^2}\hat{P}_{\alpha\alpha}\ ,\ k_\alpha = \frac{1}{\tau_F^3}\hat{k}_\alpha}$

All this follows from algebra, but also simply from dimensional analysis.

The hydrodynamic equations read

$$\boxed{\partial_{\hat{t}} n = -\partial_{\vec{q}}\cdot[\hat{\vec{u}}\,n]\ ;\ m\hat{D}_{\hat{t}}\hat{\vec{u}} = -\frac{1}{\hat{n}}\partial_{\vec{q}}\cdot\hat{P}\ ;\ \hat{D}_{\hat{t}}\hat{T} = -\frac{2}{3\hat{n}k_B}\partial_{\vec{q}}\cdot\vec{k} - \frac{2}{3nk_B}\vec{\nabla}\otimes\hat{\vec{u}} : P}$$

The zeroth order hydro then follows as before & $\hat{f}_0 = \frac{\hat{n}_0}{(2\pi m k_B \hat{T}_0)^{3/2}} e^{-\frac{\delta \hat{\vec{v}}^2}{2k_B\hat{T}_0}}$

leading to

$$\partial_{\hat{t}}\hat{n} = -\vec{\nabla}\cdot[\hat{\vec{u}}\; n]$$

$$m\hat{D}_{\hat{t}}\vec{u} = -\frac{1}{n}\vec{\nabla}\cdot[n k_B \hat{T}] \quad \Rightarrow \text{no relaxation}$$

$$\hat{D}_{\hat{t}}\hat{T} = -\frac{2\hat{T}}{3}\vec{\nabla}\cdot\hat{\vec{u}}$$

The first order now takes a nicer form